Recent Advances in Stochastic Vehicle Routing

Michel Gendreau
CIRRELT and MAGI
École Polytechnique de Montréal

SBPO 42º
Bento Gonçalves– August 30-September 3, 2010
Outline

1. Introduction and overview of the stochastic routing field
2. An exact algorithm for the m-Vehicle Routing Problem with Stochastic Demands
3. The Consistent Vehicle Routing Problem with Stochastic Customers
4. Conclusion and perspectives
Acknowledgements

- Walter Rei
  CIRRELT and ESG UQÀM

- Ola Jabali, Tom van Woensel and Ton de Kok
  School of Industrial Engineering
  Eindhoven University of Technology
Introduction and overview of the stochastic routing field
Vehicle Routing Problems

- Introduced by Dantzig and Ramser in 1959
- One of the most studied problem in the area of logistics
- The basic problem involves delivering given quantities of some product to a given set of customers using a fleet of vehicles with limited capacities.
- The objective is to determine a set of minimum-cost routes to satisfy customer demands.
Vehicle Routing Problems

Many variants involving different constraints or parameters:

- Introduction of travel and service times with route duration or time window constraints
- Multiple depots
- Multiple types of vehicles
- ...
What is Stochastic Vehicle Routing?

Basically, any vehicle routing problem in which one or several of the parameters are not deterministic:

- Demands
- Travel or service times
- Presence of customers
- ...

Advances in Stochastic Vehicle Routing
Information and decision-making

As in any other stochastic optimization problem, a key issue is:

- How do the revelation of information on the uncertain parameters and decision-making (optimization) interact?
  - When do the values taken by the uncertain parameters become known?
  - What changes can I (must I) make in my plans on the basis of new information that I obtain?
Modelling paradigms

Real-time optimization (re-optimization)

- Based on the implicit assumption that information is revealed over time as the vehicles perform their assigned routes.
- Dynamic programming and related approaches (Secomandi et al.)
- Routes are created piece by piece on the basis on the information currently available.
- Not always practical (e.g., recurrent situations)
Modelling paradigms

A priori optimization

- A solution must be determined beforehand; this solution is “confronted” to the realization of the stochastic parameters in a second step.

Approaches:

- Chance-constrained programming
- (Two-stage) stochastic programming with recourse
- Robust optimization
- [“Ad hoc” approaches]
Chance-constrained programming

- One of the two main approaches in stochastic programming.
- CCP relies on the introduction of probabilistic constraints in mathematical programs:
  - E.g., in VRP with stochastic demands,
    \[ \Pr\{\text{total demand assigned to route } r \leq \text{capacity} \} \geq 1-\alpha \]
- Probabilistic constraints can sometimes be transformed into deterministic ones (e.g., in the case above if customer demands are independent and Poisson).
- This model completely ignores what happens when things do not “turn out correctly”.

Advances in Stochastic Vehicle Routing
Robust optimization

- Here, uncertainty is represented by the fact that the uncertain parameter vector must belong to a given polyhedral set (without any probability defined).
  - E.g., in VRP with stochastic demands, having set upper and lower bounds for each demand, together with an upper bound on total demand.

- Robust optimization looks in a minimax fashion for the solution that provides the best “worst case”.

- Model may be overly pessimistic.
Stochastic programming with recourse

- Second main approach in stochastic programming
- **Recourse** is a key concept in a priori optimization
  - What must be done to “adjust” the a priori solution to the values observed for the stochastic parameters!
  - Another key issue is deciding when information on the uncertain parameters is provided to decision-makers.

- In stochastic programming approaches, one typically minimizes the cost of the a priori solution plus the expected cost of recourse actions.
Stochastic programming with recourse

- One can define multi-stage stochastic programs (i.e., information about the stochastic parameters is revealed in several steps, with several possibilities for adjusting solutions).
- In general, only two stages are considered.
- Solution methods:
  - Integer L-shaped (Laporte and Louveaux)
  - Heuristics (including metaheuristics)
- Probably closer to actual industrial practices, if recourse actions are correctly defined!
Main classes of stochastic VRPs

- VRP with stochastic demands (VRPSD)
  - A probability distribution is specified for the demand of each customer.
  - One usually assumes that demands are independent (this may not always be very realistic...).

- VRP with stochastic customers (VRPSC)
  - Each customer has a given probability of requiring a visit

- VRP with stochastic travel times (VRPSTT)
  - The travel times required to move between vertices, as well as sometimes service times, are random variables.
VRP with stochastic demands

- Probably, the most extensively studied SVRP:
  - Under the reoptimization approach (Secomandi et al.)
  - Under the a priori approach (several authors) using both the chance-constrained and the recourse models.

- Classical recourse strategy:
  - Return to depot to restore vehicle capacity
  - Does not always seem very appropriate or “intelligent”

- However, recently many authors have starting proposing more creative recourse schemes:
  - Pairing routes (Erera et al.)
  - Preventive restocking (Yang, Ballou, and Mathur)
VRP with stochastic customers

- Problem grounded in the pioneering work of Jaillet (1985) on the Probabilistic Traveling Salesman Problem (PTSP)

- At first sight, the VRPSC is of no interest under the reoptimization approach.

- Recourse action:
  - “Skip” absent customers
VRP with stochastic travel times

- The least studied, but possibly the most interesting of all SVRP variants.
- Reason: it is much more difficult than others, because delays “propagate” along a route.
- Usual recourse:
  - Pay penalties for soft time windows or overtime
- All solution approaches seem relevant, but present significant implementation challenges.
An exact algorithm for the m-Vehicle Routing Problem with Stochastic Demands
Presentation outline

- $m$-VRP with stochastic demands
  - General description
  - Model
  - Related work

- General solution approach
  - Applying the L-Shaped algorithm to stochastic VRP
  - Main challenges

- Enhancing the L-Shaped method
  - Extended Lower Bounding Functional (LBF) cuts
    - Improving the lower bound for partial routes
  - Local branching
    - LB to improve both the upper and lower bounds
    - General separation strategy

- Research perspectives
**m-VRP with stochastic demands**

- **General description**
  - A fleet of *m* capacitated vehicles must deliver (collect) demands to (from) a set of customers.
  
  \[ G(V, E) = \text{undirected graph} \]

  Where \( V = \{v_1, \ldots, v_n\} \) and \( E = \{(v_i, v_j) : v_i, v_j \in V, i < j\} \)

  \[ C = (c_{ij}) \text{ travel cost matrix} \]

  \( v_1 = \text{depot} \)

  \( D = \text{capacity of each vehicle} \)

  - Demands are revealed when vehicle arrives at a client given location \( \rightarrow \xi_i = \text{demand of client } i, \text{ where } i = 2, \ldots, n \)

  - Recourse \( \Rightarrow \) return to depot if vehicle cannot fulfill customer demand

  Extra costs !!
**m-VRP with stochastic demands (cont’d)**

**Model**

**Variables:**

\[ x_{ij} = \{0, 1\} \quad \text{for } i, j > 1 \]

\[ x_{1j} = \{0, 1, 2\} \quad \text{for } j > 1 \]

**Objective:**

\[ \min \sum_{i<j} c_{ij} x_{ij} + Q(x) \]

**Constraints:**

\[ \sum_{j=2}^{n} x_{1j} = 2m \]

\[ \sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2, \quad k = 2, \ldots, n \]

\[ \sum_{v_i, v_j \in S} x_{ij} \leq |S| - \left[ \sum_{v_i \in S} E(\xi_i)/D \right], \quad S \subset V \setminus \{v_1\}, 2 \leq |S| \leq n - 2 \]

\[ x_{ij} = \{0, 1\}, \quad i, j > 1 \]

\[ x_{1j} = \{0, 1, 2\} \quad \text{for } j > 1 \]
$m$-VRP with stochastic demands (cont’d)

- Related work
  - Gendreau, Laporte and Séguin (1995):
    - 1st implementation of the L-Shaped algorithm on the problem
  - Hjorring and Holt (1999):
    - Problem considered: 1-VRP with stochastic demands
    - New cuts $\Rightarrow$ partial routes
    - Improvement of the general lower bound
  - Laporte, Louveaux and Van hamme (2002):
    - Improvement of the general lower bound for the $m$-VRP with stochastic demands
    - Generalization of the partial route cuts
    - Greedy separation procedure for cuts
  - Christiansen and Lysgaard (2007):
    - 1st branch-and-price algorithm for the $m$-VRP with stochastic demands
General solution approach

Applying the L-Shaped algorithm to stochastic VRP

0-1 Integer L-Shaped algorithm:

- Method proposed by Laporte and Louveaux (1993)
- Variant of branch-and-cut
- Assumptions:
  - \( Q(x) \) is computable
  - There exists a finite value \( L = \text{general lower bound for the recourse function} \)

The case of the \( m \)-VRP with stochastic demands:

- Problems considered:
  - Client demands are independent
  - \( \xi_j \sim N(\mu_j, \sigma_j) \) and \( \xi_j \in (0, D), j = 2, \ldots, n \)
  - For a given route \( (v_{r_1} = v_1, v_{r_2}, \ldots, v_{r_t+1} = v_1) \)

\[
Q^{1,r} = 2 \sum_{i=2}^{t} \sum_{l=1}^{i-1} \left( \sum_{s=2}^{i-1} \xi_{r_s} \leq lD < \sum_{s=2}^{i} \xi_{r_s} \right) c_{1r_i}
\]
General solution approach (cont’d)

- Applying the L-Shaped algorithm to stochastic VRP (cont’d)

  The case of the \( m \)-VRP with stochastic demands (cont’d)

  - **Current problem**
    - Relaxation:
      - Integrality constraints
      - Subtour and capacity constraints
      - Function \( Q(x) \) is replaced by a lower bound \( \Theta \)

- Imposing subtour and capacity constraints:

  \( x \Rightarrow m \) feasible routes

  Equivalent to the deterministic case using \( \mu_i \), for \( i = 2, \ldots, n \)

  Lysgaard, Letchford and Eglese (2004):
  
  Capacity inequalities, Framed capacity inequalities, Strengthened comb inequalities, etc.
General solution approach (cont’d)

- Applying the L-Shaped algorithm to stochastic VRP (cont’d)

  The case of the \( m \)-VRP with stochastic demands (cont’d)

- Obtaining lower bound \( \Theta \):

  1. Optimality cuts
     Let \( x^r \) define the \( r \)-th feasible solution to the problem
     Define
     \[
     E^r = \{(v_i, v_j) \in E : i, j > 1 \text{ and } x^r_{ij} = 1\}
     \]
     \( \Theta_r \) = recourse cost associated with solution \( x^r \)
     Valid optimality cut obtained by:
     \[
     \Theta \geq \Theta_r \left( \sum_{(v_i, v_j) \in E^r} x_{ij} - n + m + 2 \right)
     \]

  2. LBF cuts

     Valid inequalities that are used to bound the recourse cost associated with partial solutions
General solution approach (cont’d)

Main challenges

Sources of difficulty:

- Approximation of $Q(x)$
  - Hard to obtain a good general lower bound $L$
  - Information provided by optimality cuts is very local
  - High number of LBF cuts need to be added

- Quality of the upper bound
  - No guarantees that a good set of routes will be obtained early in the solution process

- Risk of enumeration
Enhancing the L-Shaped method

- Extended LBF cuts
  - General idea
    - Partial Route

Illustration:

A series of sequences and unstructured sets of customers that are connected and that begin and end at the depot.

Denote by \( (v_{s_1}', \ldots, v_{s_l}'') \), \( v_{s_l}' \neq v_{s_l}'' \), the \( l \) sequence of customers, \( l = 1, 2, 3 \).

- Information given by the cut
  Provide a lower bound on the value of the recourse cost for partial routes.
Enhancing the L-Shaped method (cont’d)

Extended LBF cuts (cont’d)

Classical Cuts

Introduced by Hjorring and Holt (1999)

Illustration:

Let:

\[ S = \{v_1, v'_{S_1}, \ldots, v''_{S_1}\} \Rightarrow \text{Chain} \]

\[ T = \{v'_{S_3}, \ldots, v''_{S_3}, v_1\} \Rightarrow \text{Chain} \]

\[ U \Rightarrow \text{Subset such that } U \cap S = \{v''_{S_1}\} \text{ and } U \cap T = \{v'_{S_3}\} \]
Enhancing the L-Shaped method (cont’d)

- Extended LBF cuts (cont’d)
- Classical Cuts (cont’d)

Bounding the partial route

**Step 1:** Aggregate information in $U$

Define node $v_0$ such that:

- $\xi_0 = \sum_{v_i \in U \setminus \{v''_{S1}, v'_{S3}\}} \xi_i$
- $c_{10} = \min_{v_i \in U \setminus \{v''_{S1}, v'_{S3}\}} \{c_{1i}\}$

**Step 2:** Create aggregated route and evaluate

Insert node $v_0$ in the partial route:

$$(v_1, \ldots, v''_{S1}, v_0, v'_{S3}, \ldots, v_1)$$

Compute $Q(x)$ on route $\Rightarrow P_1$ lower bound on all routes that share $S$ and $T$ and where set $U$ is unsequenced.
Enhancing the L-Shaped method (cont’d)

- Extended LBF cuts (cont’d)
- LBF cuts with improved lower bounds

General idea:
Use the sequenced parts of a partial route, to improve the lower bound on the recourse cost used to produce the associated cut

Illustration:

\[ S^1 = \{ v_1, v'_S1, \ldots, v''_S1 \} \Rightarrow \text{Chain no.1} \]
\[ S^2 = \{ v'_S2, \ldots, v''_S2 \} \Rightarrow \text{Chain no.2} \]
\[ S^3 = \{ v'_S3, \ldots, v''_S3, v_1 \} \Rightarrow \text{Chain no.3} \]
\[ U^1 \Rightarrow v_0^1 \quad U^2 \Rightarrow v_0^2 \]
\[ (v_1, \ldots, v''_S1, v_0^1, v'_S2, \ldots, v''_S2, v_0^2, \ldots, v_1) \]
Compute \( P_2 \)
Enhancing the L-Shaped method (cont’d)

- Extended LBF cuts (cont’d)
  - LBF cuts that bound a larger subset of partial routes

General idea:
Use the connected components of a partial route to produce a cut that provides a lower bound for a larger subset of solutions to the current problem.

Illustration:

\[ U^k \Rightarrow \text{Subset no. } k, k = 1, 2, 3, 4, 5 \]
\[ U^k \Rightarrow v_{0_k}, k = 1, 2, 3, 4, 5 \]
\[ (v_1, v_{0_1}, v_{0_2}, v_{0_3}, v_{0_4}, v_{0_5}, v_1) \]
Compute \( P_3 \)
Enhancing the L-Shaped method (cont’d)

Extended LBF cuts (cont’d)

Obtaining the cuts:

For a given partial route $h$:
Let $S_h$ = the set of all components of a partial route considered to be chains $\Rightarrow S_h = \{S^1_h, S^2_h, \ldots\}$
Let $U_h$ = the set of all components of a partial route considered to be unsequenced subsets $\Rightarrow U_h = \{U^1_h, U^2_h, \ldots\}$
Define $R_h = S_h \cup U_h$

Set: $W_h(x, S_h, U_h) = \sum_{k=1}^{\left|S_h\right|} \sum_{(v_i, v_j) \in S^k_h} x_{ij} + \sum_{k=1}^{\left|U_h\right|} \sum_{v_i, v_j \in U^k_h} x_{ij} - |R_h| + 1$

Let $P^h$ = associated lower bound on the recourse cost

Note:
- $P^h_3 \leq P^h_1 \leq P^h_2$

Case of $r$ partial routes:
- $\Theta \geq L + (P - L) \left( \sum_{h=1}^{r} W_h(x, S_h, U_h) - r + 1 \right)$
- $P = \sum_{h=1}^{r} P^h + L_r$
Enhancing the L-Shaped method (cont’d)

Extended LBF cuts (cont’d)

Separation procedures

A solution \((x^\nu, \Theta^\nu)\) can be cut using an LBF valid inequality if the following conditions are met:

- There exists at least one \(h\) such that \(W_h(x^\nu, S_h, U_h) = 1\)
- \(P \geq \Theta^\nu\)

Procedures:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Approach</td>
<td>Greedy procedure for building chains (S) and (T) and set (U)</td>
<td>Using the graph induced by the current solution find all connected components (chains and sets)</td>
</tr>
<tr>
<td>Cuts generated</td>
<td>Classical</td>
<td>All types</td>
</tr>
<tr>
<td>Efficiency</td>
<td>Extremely fast – no guarantees</td>
<td>Slower – exact separation</td>
</tr>
</tbody>
</table>
Enhancing the L-Shaped method (cont’d)

Extended LBF cuts (cont’d)

Results

Test problems:

- Instances obtained using generator of Laporte, Louveaux and Van hamme (2002):

Let $\bar{f} = \frac{\sum_{i=2}^{n} E(\xi_i)}{mD}$ define the filling coefficient

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\bar{f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>60, 70, 80</td>
<td>90%, 92.5%, 95%</td>
</tr>
<tr>
<td>3</td>
<td>50, 60, 70</td>
<td>85%, 87.5%, 90%</td>
</tr>
<tr>
<td>4</td>
<td>40, 50, 60</td>
<td>80%, 82.5%, 85%</td>
</tr>
<tr>
<td>5</td>
<td>30, 40, 50</td>
<td>75%, 77.5%, 80%</td>
</tr>
</tbody>
</table>

- In each case, 10 instances were generated (total: 360 instances)

- Computation time of 3 hours
Enhancing the L-Shaped method (cont’d)

- Extended LBF cuts (cont’d)

- Results (cont’d)

### Heuristic separation vs. Exact separation to produce LBF1

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>74 446.06 187.64</td>
<td>0 -</td>
<td>6 1.55% 994.98</td>
<td>10 1.81% 1.67 %</td>
</tr>
<tr>
<td>3</td>
<td>52 668.79 305.45</td>
<td>0 -</td>
<td>17 1.81% 3233.16</td>
<td>21 2.51% 2.30%</td>
</tr>
<tr>
<td>4</td>
<td>30 1247.55 614.01</td>
<td>10 2031.47 4.57%</td>
<td>12 4.98% 2440.08</td>
<td>38 4.45% 3.77%</td>
</tr>
<tr>
<td>5</td>
<td>13 1622.80 957.44</td>
<td>1 3313.64 1.10%</td>
<td>6 1.60% 5620.21</td>
<td>70 4.42% 4.16%</td>
</tr>
<tr>
<td>Ave.</td>
<td>169 747.39 358.79</td>
<td>11 2148.03 4.25%</td>
<td>41 2.67% 3022.82</td>
<td>139 3.95% 3.59%</td>
</tr>
</tbody>
</table>

**Observations:**

When using the exact separation procedure:

- Better solution times for those instances solved to optimality
- 41 additional instances were solved when using exact separation
- Slightly better optimality gaps for those instances not solved to optimality
Enhancing the L-Shaped method (cont’d)

Extended LBF cuts (cont’d)

Results (cont’d)

Comparing LBF cuts

<table>
<thead>
<tr>
<th></th>
<th>LBF1</th>
<th>LBF2</th>
<th>LBF3</th>
<th>LBF1-2</th>
<th>LBF2-3</th>
<th>LBF1-2-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>nb. i. sol.</td>
<td>210</td>
<td>225</td>
<td>163</td>
<td>232</td>
<td>227</td>
<td>232</td>
</tr>
<tr>
<td>sol. times</td>
<td>878.91</td>
<td>941.70</td>
<td>679.21</td>
<td>806.51</td>
<td>1011.13</td>
<td>935.72</td>
</tr>
<tr>
<td>gaps</td>
<td>3.64%</td>
<td>3.43%</td>
<td>5.65%</td>
<td>3.42%</td>
<td>3.63%</td>
<td>3.50%</td>
</tr>
</tbody>
</table>

Observations:

- LBF2 is the best strategy when using only one type of valid inequalities
- Better results are obtained when LBF’s are applied together (LBF1-2 and LBF1-2-3)
Enhancing the L-Shaped method (cont’d)

- Local branching

  - General principles

    LB method proposed by Fischetti and Lodi (2003)

    Based on the use of generic solvers (CPLEX)

    Separation of the feasible region:

    - let \( x^0 \) = feasible solution to the original problem,
    - let \( E_0 = \{(v_i, v_j) \in E \mid x^0_{ij} = 1, v_i \neq v_1\} \),
    - let \( \kappa \) = positive integer.

    The Hamming distance:

    \[
    \Delta(x, x^0) = \sum_{(v_i, v_j) \in E_0} (1 - x_{ij}) + \sum_{(v_i, v_j) \in E \setminus E_0} x_{ij}
    \]

    \[
    \Delta(x, x^0) \leq \kappa \text{ (left branch)} \quad \lor \quad \Delta(x, x^0) \geq \kappa + 1 \text{ (right branch)}
    \]

    Use generic solver on left branch subproblem
Enhancing the L-Shaped method (cont’d)

- Local branching (cont’d)
  - LB descent to produce valid inequalities

\[ \Delta (x, x^0) \leq K \]
\[ \Delta (x, x^0) \geq K + 1 \]
\[ \Delta (x, x^1) \leq K \]
\[ \Delta (x, x^1) \leq K + 1 \]
\[ \Delta (x, x^2) \leq K \]
\[ \Delta (x, x^2) \leq K + 1 \]
Enhancing the L-Shaped method (cont’d)

Local branching (cont’d)

System of valid inequalities

**Proposition**

Let \( P_n, n = 1, \ldots, m \), define a local branching descent and \( \overline{\Theta}_n, n = 1, \ldots, m \), be valid lower bounds on the recourse value for each of the subproblems in the descent, then the following system of equations defines a set of valid inequalities for the *current problem*:

\[
\Theta \geq L + (\overline{\Theta}_n - L)w_n, \quad n = 1, \ldots, m
\]

\[
\kappa_n - \Delta(x, x^n) \leq n_1 \sum_{j=1}^{n} w_j, \quad n = 1, \ldots, m
\]

\[
\Delta(x, x^n) - \kappa_n \leq (1 - w_n)n_1, \quad n = 1, \ldots, m
\]

\[
\sum_{j=1}^{m} w_j \leq 1
\]

\[
w_n \in \{0, 1\}, \quad n = 1, \ldots, m
\]
Enhancing the L-Shaped method (cont’d)

Local branching (cont’d)

System of valid inequalities (cont’d)

System bounds the value of $Q(x)$ following the local branching descent $P_n$, $n = 1, \ldots, m$

Advantages:

- If $\overline{\Theta}_n > L$, $n = 1, \ldots, m$, then system improves the lower bound $\Theta$ in the subregions explored
- System bounds $Q(x)$ in different subregions of $X$
- Through the descent $P_n$, $n = 1, \ldots, m$, improvements to the upper bound may also be possible

Limitations:

- System of inequalities makes the current problem harder to solve
Enhancing the L-Shaped method (cont’d)

Local branching (cont’d)

First case study

Stochastic 1-VRP:
Use LB to tighten the lower bound obtained for each subproblem processed by the branch and cut algorithm

280 instances ($n=20, 30, 40, \ldots, 90$ and $f = 95\%, 97.5\%, \ldots, 110\%$)

L-S and L-S+LB for run times of 1200 sec.

Classification of instances:
176 easy (< 60 sec.) 30 moderate ([60,1200) sec.) and 74 hard (> 1200 sec.)

Observations:

- For easy instances $\Rightarrow$ implementations are equivalent
- For moderate instances $\Rightarrow$ 359.27 sec. (L-S) vs. 35.35 sec. (L-S+LB)
- 24 extra instances solved when using LB
- On instances not solved $\Rightarrow$ 3.42% (L-S) vs. 2.04%(L-S+LB)
Enhancing the L-Shaped method (cont’d)

Local branching (cont’d)

Extension to $m$-VRP with stochastic demands

General strategy based on two principles

**Principle 1**: Apply LB from feasible solutions

Advantages:
- Strengthen the lower bound around optimality cuts
- Improving the upper bound

**Principle 2**: Diversification using LB neighborhoods

Following a LB descent:
Let $\hat{x} \Rightarrow$ solution to the *current problem*

Then:
\[
\hat{x} \in \{ x \mid \Delta(x, x^n) \leq \kappa_n, n = 1, \ldots, m \}
\]

Or
\[
\hat{x} \in \{ x \mid \Delta(x, x^n) \geq \kappa_n + 1, n = 1, \ldots, m \}
\]
Enhancing the L-Shaped method (cont’d)

- Local branching (cont’d)

- Extension to $m$-VRP with stochastic demands (cont’d)
  
  General strategy based on two principles (cont’d)

  **Principle 2:** Diversification using LB neighborhoods (cont’d)

  If $\hat{x} \in \{x \mid \Delta(x, x^n) \leq \kappa_n, n = 1, \ldots, m\} \Rightarrow$ LB descent provides a lower bound on the value of $Q(\hat{x})$

  Otherwise $\Rightarrow$ LB descent does not bound $Q(\hat{x})$

  Defined strategy:
  
  - Temporarily add $\Delta(x, x^n) \geq \kappa_n + 1, n = 1, \ldots, m$ to Current problem

    Allows for diversification in the separation process

  - Only use the system of valid inequalities at the end of the separation process

    Tighten lower bound obtained for the processed subproblem

  - p.27/29
Research perspectives

- Heuristic separation for extended LBF
  Exact separation may not be efficient when applied to larger problems

- Assignment LBF cuts
  Produce LBF cuts based on the assignment (clients → vehicles) for partial solutions

- Branch-and-cut-and-price?
  Integrating both solution approaches

- Extensions to other stochastic VRP
The Consistent VRP with Stochastic Customers
Problem definition

The consistent vehicle routing problem

- First introduced by Groër, Golden, and Wasil (2009)
  - Have the same driver visiting the same customers at roughly the same time each day that these customers need service
  - Focus is on the customer
  - Planning is done for $D$ periods, known demand, $m$ vehicles
  - Arrival time variation is no more than $L$
- Minimize travel time over $D$ periods
Problem definition

The consistent vehicle routing problem with stochastic customers

- Each customer has a probability of occurring
  - Same driver visits the same customers
  - A delivery time window is quoted to the customer → (Self-imposed TW)
- Cost structure
  - Penalties for early and late arrivals
  - Travel times

*a priori* approach

- Stage 1: Plan routes and set targets
- Stage 2: Compute travel times and penalties
Problem data

- An undirected graph \( G=(V,A) \)
  - \( V=\{v_1,..,v_n\} \) is a set of vertices
  - \( E=\{(v_i, v_j): v_i, v_j \in V, i<j\} \) is a set of edges
  - Vertex \( v_i \) corresponds to the depot
  - Vertices \( v_2,..,v_n \) correspond to the potential clients
  - \( c_{ij} \) is the travel time between \( i \) and \( j \)
- \( m \) is the number of available vehicles
- A vehicle can travel at most \( \lambda \) hours
- \( p_i \) is the probability that client \( i \) places an order
- \( \Omega \) is the set of possible scenarios associated with the occurrences for all customers
Model

- First-stage decision variables:
  - $x_{ij} = 1$, if client $j$ is visited immediately after client $i$ for $2 \leq i < j \leq n$, and 0 otherwise
  - $x_{1j}$ can take the values 0, 1 or 2
  - $t_i$, target arrival time at customer $i$

- $\xi$: a random vector containing all Bernoulli random variables associated with the customers.

- For each scenario $\omega \in \Omega$, let $\xi(\omega)^T = [\xi_2(\omega), \ldots, \xi_n(\omega)]$
  - $\xi_i(\omega) = 1$, if customer $i$ is present and 0 otherwise.

- $Q(x)$: second-stage cost (recourse)
Model

$$\min_x Q(x) = E_{\xi} Q(x, \xi(\omega))$$

$$\sum_{i=1}^{n} x_{1i} = 2m$$

$$\sum_{i<k} x_{ik} + \sum_{j<k} x_{ik} \leq 2 \quad v_k \in V \setminus v_1$$

$$\sum_{i,j} x_{ij} \leq |S| - \left\lfloor \frac{l(S^+)}{\lambda} \right\rfloor \quad S \subset V \setminus v_1, 2 \leq |S| \leq n - 2$$

$$0 \leq x_{0j} \leq 2 \quad v_j \in V \setminus v_1$$

$$0 \leq x_{ij} \leq 1 \quad 1 < i < j \leq n$$

$$x_{ij} \text{ integer} \quad 1 \leq i < j \leq n$$
Reformulation the objective function:

$$\min_{x, t} \tilde{c}^T x + \tilde{Q}(x)$$

$\tilde{c}^T x$ is a lower bound on the expected travel time

Gendreau, Laporte and Séguin (1995)

And

$$\tilde{Q}(x) = Q(x) - \tilde{c}^T x$$
Model

- Assumption: early arrivals do not wait for the time window
- Evaluation of the second stage cost

$Q^{r, \delta}$: expected recourse cost corresponding to route $r$ if orientation $\delta$ is chosen

$Q_P^{r, \delta}$: total average penalties associated with time window deviations for route $r$ if orientation $\delta$ is chosen

$Q_T^r$: total average travel time for route for route $r$

$$Q^{r, \delta} = Q_T^r + Q_P^{r, \delta} \quad Q(x) = \sum_{r=1}^{m} \min\{Q^{r, 1}, Q^{r, 2}\}$$
Model

Given a route $r$, we relabel the vertices on the route according to a given orientation $\delta$ as follows:

\[(v_1 = v_{1:r}^\delta, v_{2:r}^\delta, \ldots, v_{t_r}^\delta, v_{t_r+1}^\delta = v_1)\]

$\phi^\delta(v_{i,r})$: the minimum expected penalty associated with customer $v_{i,r}^\delta$

\[Q_p^{r,\delta} = \sum_{i=1}^{t_r+1} \phi^\delta(v_{i,r})\]
Model

Setting of $t_{i_r}^\delta$ and evaluation of $\phi^\delta(v_{i_r})$

Parameters:

- $A_{i_r}^\delta$ – the collection of random events where customer $v_{i_r}^\delta$ requires a visit
- $\tau$ – half length of the time window
- $p_\omega$ – probability of $\omega \in A_{i_r}^\delta$
- $a_i^\delta(\omega)$ – arrival time at $v_{i_r}^\delta$ considering $\omega \in A_{i_r}^\delta$
- $\beta$ – late arrival penalty

Variables:

- $t_{i_r}^\delta$ – target arrival time at customer $v_{i_r}^\delta$
- $e_{i_r}^\delta(t_c)$ – early arrival at customer $v_{i_r}^\delta$ by $\omega \in A_{i_r}^\delta$
- $l_{i_r}^\delta$ – late arrival at customer $v_{i_r}^\delta$ by $\omega \in A_{i_r}^\delta$

\[
\phi^\delta(v_{i_r}) = \min \sum_{\omega \in A_{i_r}^\delta} p_\omega (e_{i_r}^\delta(\omega) + \beta l_{i_r}^\delta(\omega))
\]

s.t. 

\[
[t_{i_r}^\delta - \tau] - a_{i_r}^\delta \leq e_{i_r}^\delta(\omega) \quad \forall \omega \in A_{i_r}^\delta
\]

\[
a_{i_r}^\delta - [t_{i_r}^\delta + \tau] \leq l_{i_r}^\delta(\omega) \quad \forall \omega \in A_{i_r}^\delta
\]

\[
e_{i_r}^\delta(\omega), l_{i_r}^\delta(\omega) \geq 0 \quad \forall \omega \in A_{i_r}^\delta
\]

\[
t_{i_r}^\delta \geq 0
\]
Solution procedure

Based on the Integer 0-1 L-Shaped Method proposed by Laporte and Louveaux (1993)

- Variant of branch-and-cut
  - Assumption 1: \( Q(x) \) is computable
  - Assumption 2: There exists a finite value \( L = \) general lower bound for the recourse function.

- Operates on the current problem (CP) on each node of the search tree
  - In the VRP context, CP is relaxed:
    I. Integrality constraints
    II. Subtour elimination and route duration constraints
    III. \( c'x + Q(x) \rightarrow c'x + \theta \)
0-1 Integer L-Shaped Algorithm

List contains initial relaxed CP $\overline{z} := \infty$

Choose next pendent node

$z^* \geq \overline{z}$

yes

Fathom node

no

Introduce violated feasibility constraints and LBF

yes

Integer

no

Apply branching

no

Update $\overline{z}$

Introduce optimality cuts

Advances in Stochastic Vehicle Routing
General lower bound

\[ l = \min_{i,j} c_{ij} \quad p = \min_{\forall i} p_i \quad q = \min_{\forall i} (1 - p_i) \]

We create an auxiliary graph with all distances equal to \( l \) and all probabilities are set to \( p \) and \( q \)

\rightarrow \text{a lower bound on average travel time is } (n-1)pl

\rightarrow \text{a lower bound on the penalties associated with time}
              \text{window deviations can be determined also}
Lower bounding functionals

Introduced by Hjorring and Holt (1999) bound $Q(x)$ using partial routes

Partial route $h$ consist of:

$S = (v_0, ..., v_S)$

$T = (v_0, ..., v_T)$

$U \cap S = \{v_S\}$

$U \cap T = \{v_T\}$
Lower bounding functionals

We look for a lower bound on the recourse associated with route $h, P_h$

- Bounds on $S$ → compute exactly

- Bounds on $U$: assume each node, separately, directly succeeds $v_S \rightarrow |U_h - 2| p_{s_h} \phi^{\delta} (v_{s_h})$ → compute for each node in $U$

- Bounds on $T$: assume general sequence with $U$ as in $L$ → compute for a subset of scenarios where at most one customer is absent
Lower bounding functionals

Let \( R_h = S_h \cup T_h \cup U_h \)

Let \((v_i, v_j) \in S_h \) or \( T_h \) if \( v_i \) and \( v_j \) are consecutive in \( S_h \) and \( T_h \)

\[
W_h(x) = \sum_{(v_i,v_j) \in S_h} x_{ij} + \sum_{(v_i,v_j) \in T_h} x_{ij} + \sum_{v_i,v_j \in U_h} x_{ij} - |R_h| + 1
\]

For \( r \) partial routes the following is a valid inequality:

\[
P = \sum_{h=1}^{r} P_h \quad \theta \geq L + (P - L) \left( \sum_{h=1}^{r} W_h(x) - r + 1 \right)
\]
Preliminary results

Experimental sets:

- Vertices were generated similar to Laporte, Louveaux and van Hamme (2002)
- $p$ values are randomly generated within 0.6 and 0.9
- 20 customers with 4 vehicles or 15 with 3 vehicles
## Preliminary results

<table>
<thead>
<tr>
<th>Set</th>
<th>N</th>
<th>Initial best integer</th>
<th>Initial best node</th>
<th>Initial GAP</th>
<th>Final solution</th>
<th>Final GAP</th>
<th>Run time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>356.7</td>
<td>303.1</td>
<td>15.0%</td>
<td>332.8</td>
<td>&lt;1%</td>
<td>64</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>352.5</td>
<td>264.3</td>
<td>25.0%</td>
<td>285.2</td>
<td>&lt;1%</td>
<td>55</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>397.3</td>
<td>360.3</td>
<td>9.3%</td>
<td>388.1</td>
<td>&lt;1%</td>
<td>263</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>363.3</td>
<td>303.0</td>
<td>16.6%</td>
<td>312.8</td>
<td>&lt;1%</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>407.7</td>
<td>357.0</td>
<td>12.4%</td>
<td>393.3</td>
<td>&lt;1%</td>
<td>97</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>616.2</td>
<td>554.1</td>
<td>10.1%</td>
<td>597.2</td>
<td>&lt;1%</td>
<td>879</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>486.4</td>
<td>486.4</td>
<td>6.2%</td>
<td>461.0</td>
<td>&lt;1%</td>
<td>340</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>476.5</td>
<td>405.4</td>
<td>14.9%</td>
<td>451.3</td>
<td>&lt;1%</td>
<td>27564</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>520.0</td>
<td>414.0</td>
<td>20.4%</td>
<td>455.1</td>
<td>&lt;1%</td>
<td>64638</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>449.4</td>
<td>368.5</td>
<td>18.0%</td>
<td>397.8</td>
<td>&lt;1%</td>
<td>67501</td>
</tr>
<tr>
<td>12</td>
<td>20</td>
<td>526.9</td>
<td>475.3</td>
<td>9.8%</td>
<td>478.6</td>
<td>&lt;1%</td>
<td>2242</td>
</tr>
<tr>
<td>13</td>
<td>20</td>
<td>474.0</td>
<td>436.5</td>
<td>7.9%</td>
<td>448.0</td>
<td>&lt;1%</td>
<td>1244</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>571.9</td>
<td>472.1</td>
<td>17.5%</td>
<td>486.7</td>
<td>9.35%</td>
<td>25200</td>
</tr>
<tr>
<td>16</td>
<td>20</td>
<td>444.3</td>
<td>397.7</td>
<td>10.5%</td>
<td>416.2</td>
<td>&lt;1%</td>
<td>25200</td>
</tr>
<tr>
<td>17</td>
<td>20</td>
<td>442.5</td>
<td>390.9</td>
<td>11.7%</td>
<td>414.2</td>
<td>3.13%</td>
<td>25200</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>494.4</td>
<td>401.0</td>
<td>18.9%</td>
<td>433.1</td>
<td>3.41%</td>
<td>86400</td>
</tr>
<tr>
<td>14</td>
<td>20</td>
<td>522.8</td>
<td>449.5</td>
<td>14.0%</td>
<td>503.5</td>
<td>1.53%</td>
<td>86400</td>
</tr>
</tbody>
</table>

---

Advances in Stochastic Vehicle Routing
Future research

- A subset of customers that occur with probability 1
- Multiple partial routes
- Improving the LBF
- Improving the bound on the objective function
- Sampling approach for larger sets
Conclusions and perspectives
Conclusion and perspectives

- Stochastic vehicle routing is a rich and promising research area.
- Much work remains to be done in the area of recourse definition.
- SVRP models and solution techniques may also be useful for tackling problems that are not really stochastic, but which exhibit similar structures.
- Up to now, very little work on problems with stochastic travel and service times, while one may argue that travel or service times are uncertain in most routing problems!
- Correlation between uncertain parameters is possibly a major stumbling block in many application areas, but no one seems to work on ways to deal with it.